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Stability of developing pipe flow subjected to non-axisymmetric disturbances

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The linear stability of the developing flow of an incompressible fluid in the entrance region of a circular tube is investigated. The case of non-axisymmetric small disturbances is considered in the analysis. The main-flow velocity distribution used in the stability calculations is that from the solution of the linearized momentum equation. The eigenvalue problem consisting of the disturbance equations and the boundary conditions is solved by a direct numerical integration scheme along with an iteration procedure. An orthonormalization method is employed to remove the 'parasitic errors' inherent in the numerical integration of the coupled disturbance equations. The flow is found to be unstable to non-axisymmetric disturbances with an azimuthal wavenumber of one. Neutral-stability curves and critical Reynolds numbers at various axial locations are presented. A comparison of these results is made with those for axisymmetric disturbances reported by Huang & Chen. It is found that the first instability of the flow is due to non-axisymmetric disturbances and occurs in the entrance region of the pipe with a minimum critical Reynolds number of 19780.

1. Introduction

It is well known from analytical studies that fully developed Poiseuille flow in a circular tube is stable to axisymmetric small disturbances (see, for example, Corcos & Sellars 1959; Gill 1965; Davey & Drazin 1969, among others). The flow has also been shown analytically to be stable to all non-axisymmetric small disturbances by many investigators (Lessen, Sadler & Liu 1968; Burridge 1970; Salwen & Grosch 1972; Garg & Rouleau 1972). In addition, it has been found that the flow is least stable to the non-axisymmetric disturbance with azimuthal wavenumber $n = 1$ and is less stable to this disturbance than to an axisymmetric disturbance ($n = 0$), for both the wall and centre modes. The work of Graebel (1970), on the other hand, has shown that Poiseuille pipe flow is unstable to non-axisymmetric small disturbances at large axial wavenumbers, giving critical Reynolds numbers of the order of 10–100 for azimuthal wavenumbers of 2 and larger. As was pointed out by Salwen & Grosch (1972), this conflicting finding of Graebel is probably due to the breakdown of the approximation in his asymptotic solution, which is valid only for small axial wavenumbers and large Reynolds numbers. As a result, it is now firmly established that Poiseuille pipe flow is globally stable to all axisymmetric and non-axisymmetric small disturbances.

In the entrance region of a circular tube, the flow is hydrodynamically developing and is, therefore, essentially of the boundary-layer type near the tube wall. Thus, instability of developing flow can be expected to occur near the wall at sufficiently large Reynolds numbers. This has been found to be the case by Tatsumi (1952*a*, *b*), who studied the stability characteristics of the flow for the least stable wall-mode disturbances of axisymmetric type using an asymptotic method of solution and obtained a minimum critical Reynolds number of 9700 in the entrance region of the pipe. The analysis of Tatsumi has been found to be inadequate by Huang & Chen (1974), who recently re-examined the stability characteristics of the same flow by a numerical method of solution. The last two authors have presented neutral-stability results and critical Reynolds numbers that are believed to be more accurate and reliable than those of Tatsumi and found a minimum critical Reynolds number of 19900 for axisymmetric disturbances in the entrance region of the pipe.

In summary, the studies of the linear stability of flow in a circular pipe so far have concluded that (i) Poiseuille flow is stable to all axisymmetric and non-axisymmetric small disturbances, (ii) Poiseuille flow is less stable to non-axisymmetric small disturbances with azimuthal wavenumber $n = 1$ than to axisymmetric small disturbances ($n = 0$) and (iii) developing pipe flow is unstable to axisymmetric small disturbances. The question that needs to be answered is the following. Is the first instability of the developing pipe flow due to axisymmetric disturbances or due to non-axisymmetric disturbances with $n = 1$? This motivated the present investigation.

The present study deals with the linear stability of developing laminar pipe flow subjected to non-axisymmetric disturbances with azimuthal wavenumber $n = 1$. The case of timewise or temporal stability of the flow is considered in the analysis. For fully developed flow, it has been found (Lessen *et al.* 1968; Burridge 1970) that, for a fixed axial wavenumber, the least stable wall mode and the least stable mode near the pipe centre exhibit a stability characteristic that has almost the same amplification rate at high Reynolds numbers. In the experimental work of Fox, Lessen & Bhat (1968), the peak amplitude of the disturbance was observed to move away from the centre of the pipe as it propagated downstream when the disturbance showed amplification. This observation seems to be in agreement with the finding of Chen (1969) that the wall mode plays a more important role than the centre mode in the nonlinear instability of the flow. In addition, since the main flow in the development region of a circular tube is of the boundary-layer type, instability of the flow, if it exists, should originate near the tube wall as it does in boundary-layer flows. For these reasons and for comparison with the results of Huang & Chen for axisymmetric disturbances, the wall mode is investigated in the present study.

The governing equations for the disturbances and the boundary conditions constitute an eigenvalue problem which is solved by a direct numerical integration scheme along with an iteration technique. Neutral-stability curves at different axial locations in the entrance region of the pipe are generated and the critical

Reynolds numbers are determined. The stability results from the present analysis are compared with those of Huang & Chen for axisymmetric disturbances.

2. Formulation of the problem

Before proceeding to the stability problem, attention is first given to the main flow. Consider a circular tube of radius r_0, with $\bar{x}$ and $\bar{r}$ denoting, respectively, the axial and radial co-ordinates ($0 \leqslant \bar{r} \leqslant r_0$). An incompressible constant-property fluid enters the tube at $\bar{x} = 0$ with a uniform velocity distribution over the cross-section. The main flow is assumed to be laminar and steady. Of the various approximate methods of solution available in the literature for flow in the entrance region of a circular tube, the solution given by Sparrow, Lin & Lundgren (1964) appears to provide the most accurate and complete velocity field.

By linearizing the inertia terms in the axial momentum equation, Sparrow *et al.* obtained a main-flow velocity field that is continuous over the cross-section and along the length of the tube from the entrance to the fully developed flow region. It is given by

$$U = 2(1-r^2) + \sum_{i=1}^{\infty} \frac{4}{\alpha_i^2}\left[\frac{J_0(\alpha_i r)}{J_0(\alpha_i)} - 1\right]\exp(-\alpha_i^2 X^*), \tag{1}$$

in which the α_i are the positive roots of

$$J_1(\alpha_i) = \tfrac{1}{2}\alpha_i J_0(\alpha_i), \tag{2}$$

with J_0 and J_1 denoting, respectively, Bessel functions of the zeroth and first order of the first kind. The dimensionless variables are

$$U = \frac{u}{\bar{u}}, \quad r = \frac{\bar{r}}{r_0}, \quad X^* = \frac{\bar{x}^*/r_0}{\bar{u}r_0/\nu}, \quad X = \frac{\bar{x}/r_0}{\bar{u}r_0/\nu}, \tag{3}$$

where u and $\bar{u}$ are, respectively, the local axial velocity of the main flow and its average value, $\bar{x}^*$ is a stretched axial co-ordinate and ν is the kinematic viscosity. The stretched axial co-ordinate $\bar{x}^*$ is related to the physical axial co-ordinate $\bar{x}$ by the relation

$$X = \int_0^{X^*} \epsilon(X^*)\, dX^*, \tag{4}$$

where ϵ is the stretching factor given by the expression

$$\epsilon(X^*) = \left[\int_0^1 (2U - \tfrac{3}{2}U^2)\frac{\partial U}{\partial X^*} r\, dr\right] \Big/ \left[\left(\frac{\partial U}{\partial r}\right)_1 + \int_0^1 \left(\frac{\partial U}{\partial r}\right)^2 r\, dr\right]. \tag{5}$$

The foregoing equations fully specify the main-flow velocity U as a function of $x = \bar{x}/r_0$ and r.

The use of the main-flow velocity given by (1) is of advantage in the stability calculations, because the velocity U and its derivatives are continuous functions of x and r and can be expressed with great accuracy.

The stability problem will now be formulated. The starting-point of the analysis is the continuity equation and Navier–Stokes equations for incompressible three-dimensional time-dependent motion. Consider a pipe flow with velocity components $(\hat{u}, \hat{v}, \hat{w})$ in the (x, r, θ) directions and with static pressure distribution $\hat{p}$. In accordance with linear stability theory, the main flow in the hydrodynamic development region of a pipe may be regarded as a parallel flow. This model is consistent with the condition that $\alpha R \gg 1$ in the stability problem. Thus, the main-flow quantities can be approximated by $(U(r), 0, 0; P(x, r, \theta))$. If $(u^+, v^+, w^+; p^+)$ are the corresponding disturbances, which depend on (x, r, θ, t), where t is the dimensionless time based on r_0 and $\bar{u}$, then the resultant flow field is given by

$$\hat{u} = U + u^+, \quad \hat{v} = v^+, \quad \hat{w} = w^+, \quad \hat{p} = P + p^+. \tag{6}$$

Substitution of (6) into the continuity equation and the Navier–Stokes equations, followed by subtraction of the main flow and neglect of squares of disturbance quantities, will lead to the linearized disturbance equations. Next, the disturbances are assumed to be of the form

$$\begin{pmatrix} u^+ \\ v^+ \\ w^+ \\ p^+ \end{pmatrix} = \begin{pmatrix} u(r) \\ v(r) \\ w(r) \\ p(r) \end{pmatrix} \exp\left[i(\alpha x + n\theta - \alpha c t)\right], \tag{7}$$

where $u(r)$, $v(r)$, $w(r)$ and $p(r)$ are the r-dependent amplitude functions, α is the axial wavenumber, n is the azimuthal wavenumber and $c = c_r + ic_i$ is the complex phase velocity. The flow is stable, neutrally stable or unstable depending on whether c_i is negative, zero or positive. Substituting (7) into the disturbance equations, one finally obtains the following equations for the dimensionless amplitude functions:

$$i\alpha u + v' + v/r + in\, w/r = 0, \tag{8}$$

$$i\alpha R(U - c)\, u + RU'v = -i\alpha Rp + u'' + u'/r - (n^2/r^2 + \alpha^2)\, u, \tag{9}$$

$$i\alpha R(U - c)\, v = -Rp' + v'' + \frac{v'}{r} - \left(\frac{n^2+1}{r^2} + \alpha^2\right) v - 2\, in \frac{w}{r^2}, \tag{10}$$

$$i\alpha R(U - c)\, w = -in\, R\frac{p}{r} + w'' + \frac{w'}{r} - \left(\frac{n^2+1}{r^2} + \alpha^2\right) w + 2\, in \frac{v}{r^2}, \tag{11}$$

where $R = \bar{u} r_0/\nu$ is the Reynolds number and the primes denote differentiation with respect to r.

The physical conditions to be satisfied by (8)–(11) at the centre of the pipe are that the disturbance fluid velocities and pressure must be bounded and continuous at $r = 0$ (see Batchelor & Gill 1962). The conditions to be satisfied at the pipe wall $r = 1$ are that the disturbance velocities vanish. Thus, the boundary conditions are

$$\left.\begin{array}{lll} u = p = 0 & \text{for} & n \neq 0 \\ v = w = 0 & \text{for} & n \neq 1 \\ v + iw = 0 & \text{for} & n = 1 \end{array}\right\} \quad \text{at} \quad r = 0, \tag{12a}$$

$$u = v = w = 0 \quad \text{for all} \quad n \text{ at } r = 1. \tag{12b}$$

To facilitate the numerical solution of the system (8)–(12), it is convenient to transform (8)–(11) into the following coupled equations by eliminating the pressure terms (Burridge 1970):

$$L^2\phi - 2\alpha n L\Omega = i\alpha R[(U-c)\,L\phi - \phi(L+\alpha^2+n^2/\alpha^2)\,U], \tag{13}$$

$$L_1\,\Omega + \frac{2\alpha n}{(n^2+\alpha^2 r^2)^2}L\phi = i\alpha R\left[(U-c)\,\Omega - \frac{nU'\phi}{\alpha r(n^2+\alpha^2r^2)}\right], \tag{14}$$

with
$$\phi = -irv, \quad \Omega = (\alpha rw - nu)/(n^2+\alpha^2r^2). \tag{15}$$

The operators are defined by

$$L \equiv \frac{d^2}{dr^2} + \frac{n^2-\alpha^2r^2}{n^2+\alpha^2r^2}\,\frac{1}{r}\,\frac{d}{dr} - \left(\alpha^2+\frac{n^2}{r^2}\right), \tag{16}$$

$$L_1 \equiv \frac{d^2}{dr^2} + \frac{n^2+3\alpha^2r^2}{n^2+\alpha^2r^2}\,\frac{1}{r}\,\frac{d}{dr} - \left(\alpha^2+\frac{n^2}{r^2}\right). \tag{17}$$

The boundary conditions (12) become, in terms of ϕ and Ω,

$$\phi = \lim_{r\to 0} r^{2-n}\phi' = \Omega = 0 \quad \text{for} \quad n \neq 0 \quad \text{at} \quad r=0, \tag{18a}$$

$$\phi = \phi' = \Omega = 0 \quad \text{for all} \quad n \text{ at } r=1. \tag{18b}$$

The coupled linear equations (13) and (14) and the boundary conditions (18) constitute a homogeneous system and thus an eigenvalue problem of the form

$$E(n,\alpha,R,c) = 0, \tag{19}$$

which when solved gives a relationship among α, R, c and n. Generally, the value of c satisfying (19) is sought as the eigenvalue for given values of n, α and R. In the present study, only the case $n = 1$ is treated numerically.

3. Numerical method of solution

The mathematical system consisting of (13), (14) and (18) was solved for the case of $n = 1$ by a direct fourth-order Runge–Kutta integration scheme along with an iteration technique to find the eigenvalues. To use the fourth-order Runge–Kutta integration scheme for the solution of a mathematical system involving a differential equation of order n, the system must be transformed into an initial-value problem in which the values of the function and its derivatives up to the $(n-1)$th are initially specified. Since (13) and (14) are of fourth order in ϕ and of second order in Ω, one needs to specify ϕ, ϕ', ϕ'', ϕ''', Ω and Ω' at $r = 0$. In addition, since $r = 0$ is a regular singular point of (13) and (14), their solutions have to be started near the point $r = 0$ with a series (e.g. Frobenius series) expansion around that point. This series expansion along with application of boundary conditions (18a) provides three sets of independent solutions. Numerical integration of (13) and (14) is then performed with proper starting values near $r = 0$ and continued to the tube wall ($r = 1$). Application of boundary conditions (18b) leads, for a non-trivial solution, to the relationship expressed by (19). The eigenvalues were obtained by employing either the iterative procedure of Muller (1956) or the differential-correction method. The latter is par-

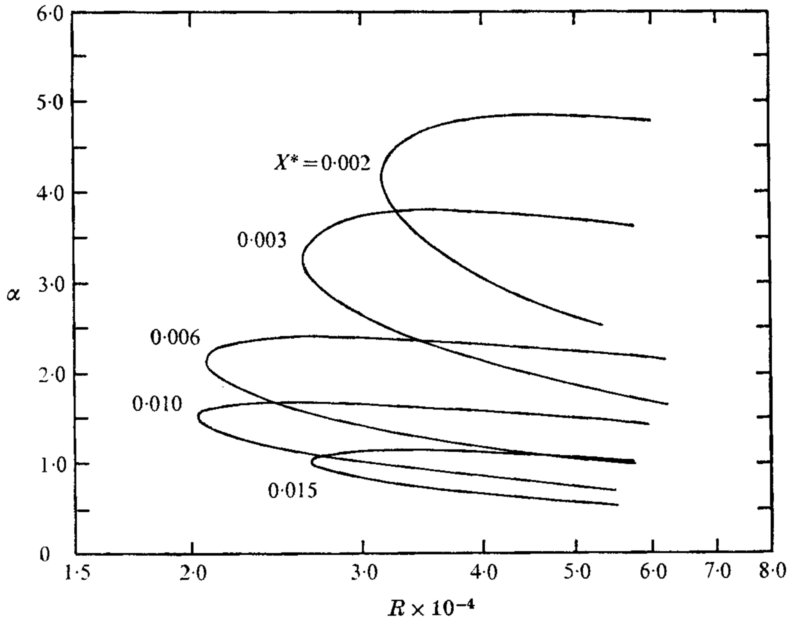

FIGURE 1. Representative neutral-stability curves, $n = 1$.

ticularly useful in mapping out the neutral-stability curves, for which $c_i = 0$. Once the eigenvalue problem has been solved, the eigenfunction can be evaluated.

It must be pointed out that instability of the flow occurs at high Reynolds numbers. Thus, (13) and (14) become ill behaved during the numerical integration for large values of αR and at least one of the three independent sets of solutions grows very rapidly. This 'parasitic error' inherent in the numerical integration causes the independent solutions to lose their characteristics and become dependent. To remove the 'parasitic error', the Gram–Schmidt orthonormalization procedure (Wazzan, Okamura & Smith 1967) was employed.

The details of the numerical procedure employed in the solution of the eigenvalue problem can be found in Huang (1973).

4. The neutral-stability results

The neutral-stability results were obtained for the least stable wall mode with azimuthal wavenumber $n = 1$. The calculations were done using single-precision arithmetic on an IBM 360/50 digital computer. These results will now be presented.

Figure 1 shows the representative neutral-stability curves at axial locations $X^* = 0{\cdot}002$, $0{\cdot}003$, $0{\cdot}006$, $0{\cdot}010$ and $0{\cdot}015$. It is seen from the figure that the neutral curves shift to the left as X^* increases from the pipe inlet and then shift back to the right as X^* increases further. That is, the flow becomes more and more unstable as X^* increases, reaches the least stable condition, and then becomes more and more stable as X^* increases further downstream.

A comparison of the neutral-stability curves at few axial locations from the present study is made with those from the axisymmetric case ($n = 0$) in figure 2.

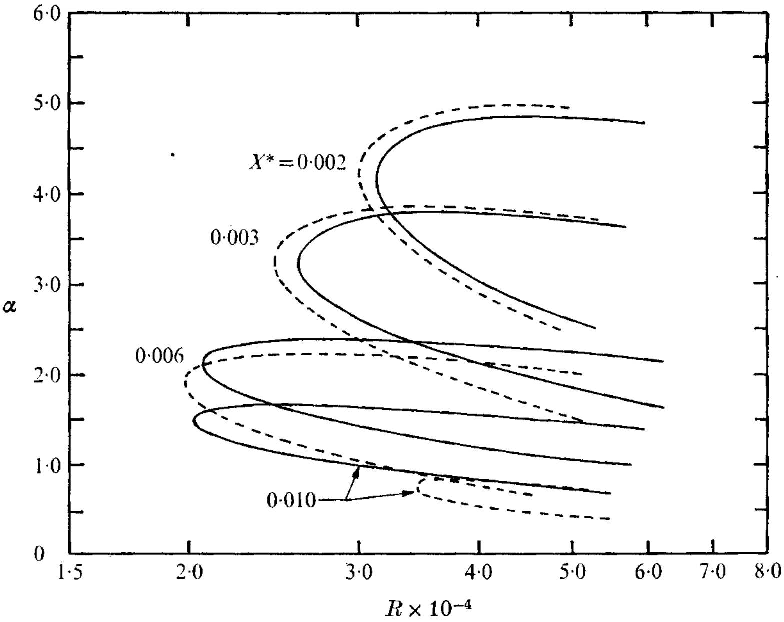

FIGURE 2. Comparison of neutral-stability curves. ——, non-axisymmetric case, $n = 1$; – – –, axisymmetric case, $n = 0$.

The latter results are from Huang & Chen (1974). An inspection of the figure reveals that axisymmetric disturbances cause a less stable flow than non-axisymmetric disturbances at small X^* values. The opposite trend is in evidence for large X^* values.

The axial variation of the critical Reynolds number R_c is shown in figure 3 as a solid line, with the physical axial co-ordinate X used as the abscissa. Included also is the (dashed) curve for axisymmetric disturbances (Huang & Chen 1974). It is of interest to note that the two curves have a somewhat similar shape. For both the axisymmetric case ($n = 0$) and the non-axisymmetric case with $n = 1$, the critical Reynolds number decreases with increasing X, attains a minimum value and then increases as X increases further downstream. The non-axisymmetric case gives a minimum critical Reynolds number $R_c = 19\,780$ at $X = 0{\cdot}00490$ as compared with the minimum value of $R_c = 19\,900$ at $X = 0{\cdot}00325$ for the axisymmetric case. For X values smaller than 0·0038, the non-axisymmetric disturbances induced a more stable flow than the axisymmetric disturbances. The opposite trend is true when X is larger than 0·0038. The fact that the flow is less stable to non-axisymmetric disturbances for large X values agrees with the conclusion of Lessen *et al.* (1968) and Burridge (1970) for fully developed Poiseuille flow. The critical stability characteristics at various axial locations for $n = 1$ are shown in table 1. In the table, α_c, R_c and $(c_r)_c$ are, respectively, the critical wavenumber, the critical Reynolds number and the critical phase speed. The last column lists the number of steps N used in the calculations.

The reason why axisymmetric disturbances are less stable than non-axisymmetric disturbances in the region close to the pipe inlet is believed to be due to the

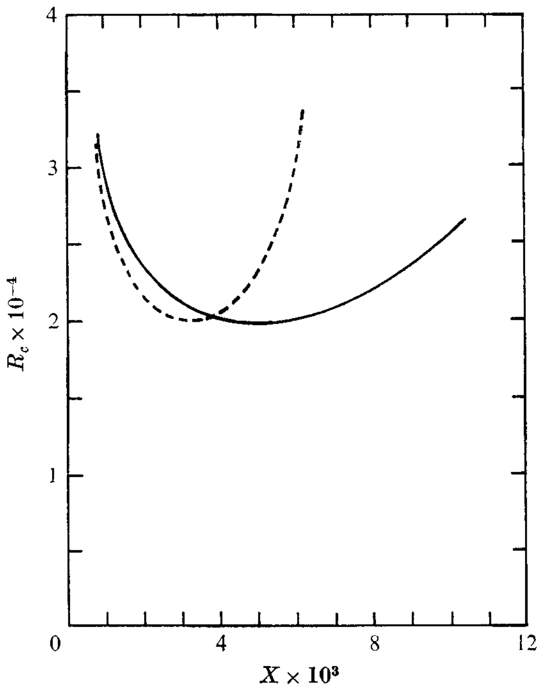

FIGURE 3. Axial variation of the critical Reynolds number. ——, non-axisymmetric case, $n = 1$; – – –, axisymmetric case, $n = 0$.

X^*	X	α_c	R_c	$(c_r)_c$	N
0·002	0·0009	4·135	31510	0·3633	200
0·003	0·0014	3·240	26220	0·3740	200
0·005	0·0026	2·363	21780	0·3869	150
0·006	0·0032	2·103	20700	0·3910	150
0·007	0·0039	1·895	20100	0·3936	150
0·009	0·0054	1·593	19840	0·3957	150
0·010	0·0062	1·475	20200	0·3949	100
0·015	0·0104	1·025	26420	0·3791	100

TABLE 1. Critical stability characteristics for non-axisymmetric disturbances ($n = 1$)

boundary-layer effect in that region. In the region very near the pipe inlet, the boundary layer is just developing along the pipe wall and the boundary-layer thickness is very small. Thus, the boundary-layer-type flow in this region is very close to being a plane flow and Squire's (1933) theorem for plane parallel flow applies.

The eigenfunctions $u(r)$, $v(r)$ and $w(r)$ for $n = 1$ with $\alpha = 2{\cdot}0$, $R = 20942$, $c_r = 0{\cdot}3884$ and $c_i = 0$ at the axial location $X = 0{\cdot}00323$ ($X^* = 0{\cdot}006$) are illustrated in figure 4. All the quantities are normalized by the maximum magnitude of v_r, the real part of v. It is noted that, for this axial location, the point

$$(\alpha, R) = (2{\cdot}0, 20942)$$

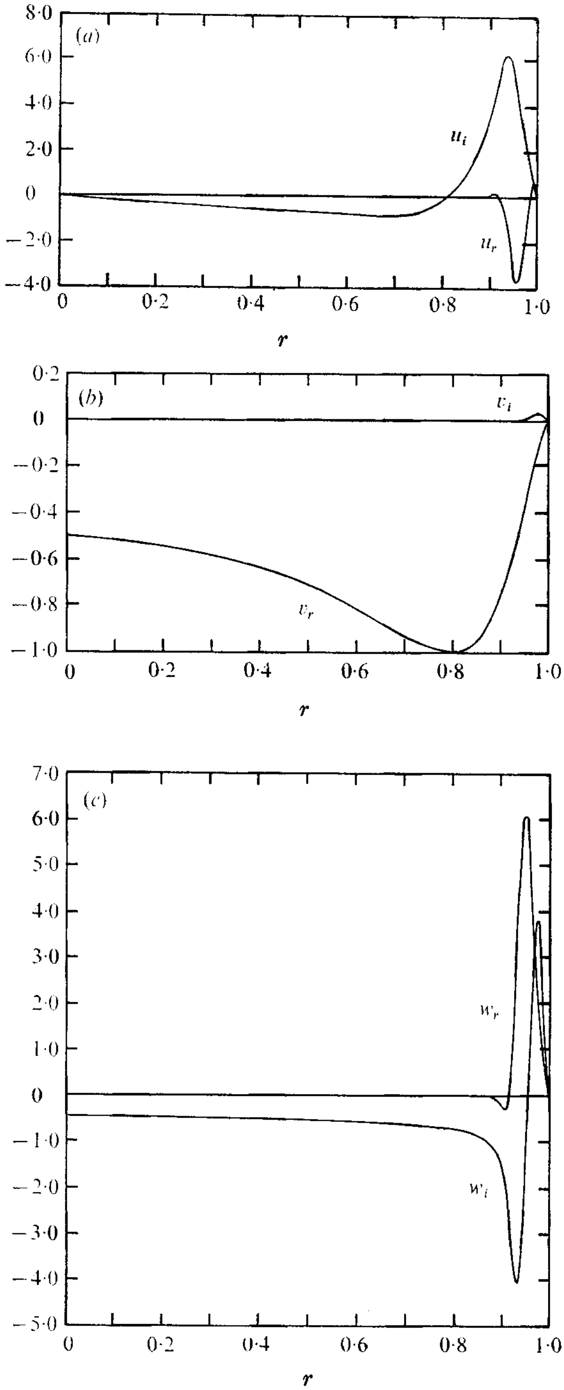

FIGURE 4. Eigenfunction at $X = 0{\cdot}00323$ ($X^* = 0{\cdot}006$) for $n = 1$, $\alpha = 2{\cdot}0$, $R = 20942$, $c_r = 0{\cdot}3884$ and $c_i = 0$. (*a*) Axial component. (*b*) Radial component. (*c*) Azimuthal component.

lies below the minimum critical point $(\alpha_c, R_c) = (2{\cdot}103, 20700)$ on the lower branch of the neutral-stability curve.

5. Conclusion

From the present study, the developing flow in the entrance region of a circular tube has been found to be unstable to non-axisymmetric small disturbances with an azimuthal wavenumber of one. The minimum critical Reynolds number of 19780 occurs at the axial location $X = 0{\cdot}0049$. This critical Reynolds number is somewhat lower than the minimum value of 19900 for the case of axisymmetric disturbances. The developing pipe flow is more stable to non-axisymmetric disturbances than to axisymmetric disturbances in the region very near the pipe inlet and less stable in the region away from the entrance. It is concluded from the present results that the first instability of pipe flow is due to non-axisymmetric disturbances and occurs in the hydrodynamic development region of the pipe.

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REFERENCES

Batchelor, G. K. & Gill, A. E. 1962 Analysis of the stability of axisymmetric jets. *J. Fluid Mech.* **14**, 529.

Burridge, D. M. 1970 The stability of Poiseuille pipe flow to non-axisymmetric disturbances. *Geophys. Fluid Dyn. Inst., Florida State University, Tech. Rep.* no. 34.

Chen, B. H. P. 1969 Finite amplitude disturbances in the stability of pipe Poiseuille flow. Ph.D. thesis, University of Rochester.

Corcos, G. M. & Sellars, J. R. 1959 On the stability of fully developed flow in a pipe. *J. Fluid Mech.* **5**, 97.

Davey, A. & Drazin, P. G. 1969 The stability of Poiseuille flow in a pipe. *J. Fluid Mech.* **36**, 209.

Fox, J. A., Lessen, M. & Bhat, W. V. 1968 Experimental investigation of Hagen–Poiseuille flow. *Phys. Fluids*, **11**, 1.

Garg, V. K. & Rouleau, W. T. 1972 Linear spatial stability of pipe Poiseuille flow. *J. Fluid Mech.* **54**, 113.

Gill, A. E. 1965 On the behaviour of small disturbances to Poiseuille flow in a circular pipe. *J. Fluid Mech.* **21**, 145.

Graebel, W. P. 1970 The stability of pipe flow. Part 1. Asymptotic analysis for small wave-numbers. *J. Fluid Mech.* **43**, 279.

Huang, L. M. 1973 Stability of the developing laminar flow in a circular tube. Ph.D. thesis, University of Missouri-Rolla.

Huang, L. M. & Chen, T. S. 1974 Stability of the developing laminar pipe flow. *Phys. Fluids*, in press.

Lessen, M., Sadler, S. G. & Liu, T. Y. 1968 Stability of pipe Poiseuille flow. *Phys. Fluids*, **11**, 1404.

Muller, D. E. 1956 A method for solving algebraic equations using an automatic computer. *Math. Tables & Aids to Comp.* **10**, 208.

Salwen, H. & Grosch, C. E. 1972 The stability of Poiseuille flow in a pipe of circular cross-section. *J. Fluid Mech.* **54**, 93.

SPARROW, E. M., LIN, S. H. & LUNDGREN, T. S. 1964 Flow development in the hydrodynamic entrance region of tubes and ducts. *Phys. Fluids*, **7**, 338.

SQUIRE, H. B. 1933 On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls. *Proc. Roy. Soc.* A **142**, 621.

TATSUMI, T. 1952*a* Stability of the laminar inlet-flow prior to the formation of Poiseuille regime. I. *J. Phys. Soc. Japan* **7**, 489.

TATSUMI, T. 1952*b* Stability of the laminar inlet-flow prior to the formation of Poiseuille regime. II. *J. Phys. Soc. Japan*, **7**, 495.

WAZZAN, A. R., OKAMURA, T. T. & SMITH, A. M. O. 1967 Stability of laminar boundary layer at separation. *Phys. Fluids*, **10**, 2540.